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ON FREE PERTURBATIONS IN HYPERSONIC LAMINAR FLOW BEHIND A PROFILE*

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Plane-parallel laminar hypersonic flow at large distances behind a wing of infinite span is considered. Non-symmetric free perturbations in the basic flow, described in terms of a blast analogy, are studied. The motion of the gas obeys the Navier-Stokes equations and is specified using two-term asymptotic representations. The symmetric and antisymmetric perturbations of the blast solution have an oscillatory form, with amplitude and frequency decaying in the downstream direction.

1. Formulation of the problem. We shall study a plane parallel flow of a hypersonic $p_\infty = 0$ real gas past a profile. The viscosity λ and thermal conductivity k are assumed to be proportional to the specific enthalpy, and we denote the corresponding proportionality coefficients by λ_0, k_0 . The ratio κ of the specific heats C_p and C_v will be assumed to be constant and to satisfy the inequality $1 < \kappa < 2$. We shall use the density ρ_∞ of the incoming flow, its velocity U_∞ and the coefficient λ_0 as the basic unit measures. The Prandtl number $Pr = C_p \lambda_0 / k_0$.

We introduce the notation $1 + v_x, v_y$ for the components of the velocity vector along the x, y axes of a Cartesian system of coordinates whose origin coincides with the streamlined profile whose abscissa axis coincides with the direction of the incoming flow. We denote the pressure, density and specific enthalpy by p, ρ, w respectively. In describing the motion of gas we shall use the system of Navier-Stokes equations and the Mises x, Ψ variables.

The principal terms of the asymptotic expansions $x \rightarrow \infty$ describing the laminar hypersonic flow of a viscous, heat conducting gas behind a body of finite dimensions, were obtained in [1]. The solution constructed there is symmetrical about the streamline $\Psi = 0$ and includes

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two regions with essentially different properties, the outer flow and the laminar wake. The solution of /1/ holds only to a first approximation, and is, in fact, perturbed everywhere due to various physical factors such as heat emission or the injection of gaseous mass from the surface of the streamlined body /2/, the action of lift /3/, etc. (in the case of an inviscid isentropic flow the perturbed blast solution can describe flow past a half-body /4/. In addition to these, perturbation of the basic solution /1/ can be caused by the specific properties of the hypersonic flow near a specific body. The perturbations, which are called free, satisfy the Rankine-Hugoniot conditions at the discontinuity and the matching conditions at the centre of the wake. The purpose of this paper is to construct such perturbations.

Since the equations describing the flow to a second approximation are linear, it is convenient to separate the arbitrary perturbations of the basic flow into symmetric and anti-symmetric perturbations, and study them separately. We shall call symmetric the perturbations in which the correction terms in the expansions for v_x, p, ρ, w are even and for v_y, y are odd functions of the variable Ψ , otherwise we shall speak of antisymmetric perturbations (i.e. perturbations in which the corrections in the expansions of the first four parameters are odd functions, and of the last two are even functions of Ψ).

We shall solve the problem of finding the free perturbations as an inverse problem, i.e. we shall specify a perturbation of the shock front belonging to some class of functions and construct the solution, and then determine the specific form of this perturbation using the symmetry conditions at the wake centre.

2. Outer region. It was shown in /5/, while studying the problem of supersonic flow past a wedge with a weakly curved surface, that the flow is formed in the course of successive reflections of the perturbations from the shock wave and the side of the wedge. As was shown in /6/, in a flow past a blunted wedge the perturbations decay when $x \rightarrow \infty$ as $\text{Re}(C_2 x^z)$, where C_2 and z are complex constants. In this connection we shall seek, following /7/, the perturbations of the outer flow in the class of complex powers, specifying the form of the shock wave in terms of an expansion (henceforth, all expansions will be written for the half-plane $\Psi \geq 0$ and the sign of the real part Re in the correction terms will be omitted)

$$y_s = Cx^{1/2} (1 + C_2 x^z + \dots) \quad (2.1)$$

We make the constant z obey the inequality $-1 < \text{Re } z < 0$ whose first part ensures the inviscid character of the flow in the outer region /8/, and the second part the decay of the perturbations as $x \rightarrow \infty$.

The outer expansions of the flow parameters corresponding to (2.1) are given by the formulas

$$\begin{aligned} v_x &= v_{x0} x^{-1/2} (v_{x11} + C_2 x^z v_{x12} + \dots) \\ v_y &= v_{y0} x^{-1/2} (v_{y11} + C_2 x^z v_{y12} + \dots) \\ p &= p_0 x^{-1/2} (p_{11} + C_2 x^z p_{12} + \dots) \\ \rho &= \rho_0 (\rho_{11} + C_2 x^z \rho_{12} + \dots) \\ w &= w_0 x^{-1/2} (w_{11} + C_2 x^z w_{12} + \dots) \\ y &= Cx^{1/2} (y_{11} + C_2 x^z y_{12} + \dots) \end{aligned} \quad (2.2)$$

The quantities with indices 11 and 12 in these expansions are functions of the self-modelling variable $\eta = \Psi x^{-1/2}/C$ and the normalizing coefficients are found from the relations

$$\begin{aligned} p_0 &= -v_{x0} = \frac{8C^2}{9(\kappa+1)}, \quad v_{y0} = \frac{4C}{3(\kappa+1)} \\ \rho_0 &= \frac{\kappa+1}{\kappa-1}, \quad w_0 = \frac{8\kappa C^2}{9(\kappa+1)^2} \end{aligned}$$

The first-approximation functions are known by virtue of the analogy with the one-dimensional isenergetic motion /9/, from the exact solution of the problem of strong explosion /10/ in Lagrangian variables.

Substituting (2.2) into the Navier-Stokes equations, we obtain the following linear second-approximation system:

$$\left(\frac{3}{2}z + 1\right)y_{12} - \eta \frac{dy_{12}}{d\eta} = \frac{2}{\kappa+1}v_{y12}, \quad w_{12} = \frac{\kappa+1}{\kappa-1}p_{11} \frac{dy_{12}}{d\eta} + \frac{p_{12}}{\rho_{11}}, \quad \left(\frac{3}{2}z - \frac{1}{2}\right)v_{y12} - \eta \frac{dv_{y12}}{d\eta} + \frac{dp_{12}}{d\eta} = 0 \quad (2.3)$$

$$\frac{p_{12}}{\rho_{11}} - \kappa \frac{p_{12}}{\rho_{11}} = 3(z+1)\eta^{3z/2}, \quad p_{12} = \rho_{11}w_{12} + \rho_{12}w_{11}, \quad v_{x12} = \frac{1}{\kappa+1}(2v_{y12}v_{y11} + \kappa w_{12})$$

The system of ordinary differential equations (2.3) is of second order, since the first two equations can yield another algebraic equation connecting the second-approximation functions. The Cauchy data for system (2.3) follow from the linearized Rankine-Hugoniot relations for the shock wave (2.1)

$$p_{12}(1) = 3\left(z + \frac{1}{\kappa + 1}\right), \quad y_{12}(1) = \frac{2}{\kappa + 1} \tag{2.4}$$

Before constructing the solution in the region of the wake, we shall write the asymptotic expansions for the first and second approximation functions as $\eta \rightarrow 0$

$$\begin{aligned} y_{11} &= c_1 \eta^{(\kappa-1)/\kappa} + \dots, \quad p_{11} = p_{21} + \dots, \quad c_1 = \frac{\kappa}{\kappa + 1} p_{21}^{-1/\kappa} \\ y_{12} &= c_2 \eta^2 v + c_y + \dots, \quad p_{12} = c_3 \eta^{2y+1} + c_p + \dots, \\ z_y &= \frac{3}{2} z + \frac{\kappa - 1}{\kappa} \\ c_2 &= \frac{6(\kappa - 1)(z + 1)}{(\kappa + 1)(3\kappa z + 2\kappa - 2)} p_{21}^{-1/\kappa}, \quad c_3 = \frac{(\kappa + 1)(3\kappa - 2)}{2\kappa(3\kappa z + 4\kappa - 2)} c_2 \end{aligned} \tag{2.5}$$

Here p_{21} is a constant known from the exact solution of the problem of strong explosion of a plane charge /10/, and the constants $c_y(z, \kappa)$ and $c_p(z, \kappa)$ can be found by numerical integration of problem (2.3), (2.4).

3. Laminar wake. The outer expansions cease to hold in the neighbourhood of $\eta = 0$, since the gradients of the flow parameters are here so large, that the thermal conductivity and viscosity begin to play a decisive role. Following /1/ we substitute the asymptotic representations (2.5) into the expansions (2.2) and pass to the inner variable $\zeta = \Psi x^{1/4}/C$. As a result we obtain the asymptotic expansions of the flow parameters in the matching region $\eta \ll 1, \zeta \gg 1$

$$\begin{aligned} p &= p_0 x^{-1/4} (p_{21} + C_2 c_3 x^{3/4} \zeta^2 v^{+1} + C_2 c_p x^2 + \dots) \\ y &= C x^{(\kappa+3)/6\kappa} [c_{10} \zeta^{(\kappa-1)/\kappa} + C_2 c_2 x^{1/4} \zeta^2 v + C_2 c_y x^2 + \dots] \\ z_1 &= \frac{z}{4} - \frac{2\kappa - 1}{2\kappa}, \quad z_2 = z + \frac{\kappa - 1}{2\kappa} \end{aligned} \tag{3.1}$$

The limit expansions (3.1) make it possible to determine the form of the inner expansions describing the flow in the region of a laminar wake

$$\begin{aligned} p &= p_0 x^{-1/4} (p_{21} + C_2 x^{3/4} p_{22} + C_2 x^2 p_{23} + \dots) \\ y &= C x^{(\kappa+3)/6\kappa} (y_{21} + C_2 x^{1/4} y_{22} + C_2 x^2 y_{23} + \dots) \end{aligned}$$

and also yield the asymptotic conditions of matching with the outer region (the indices 21, 22 and 23 denote the functions of the selfsimilar variable ζ). Substituting the inner expansions into the system of Navier-Stokes equations, we obtain the equations for the third-approximation functions $dp_{23}/d\zeta = dy_{23}/d\zeta = 0$ and this, taking the matching conditions $p_{23} = c_p + \dots, y_{23} = c_y + \dots, \zeta \rightarrow \infty$ into account, yields

$$p_{23} = c_p, \quad y_{23} = c_y \tag{3.2}$$

Next we write the asymptotic expansions for all the flow parameters, retaining only the correction terms of order $x^{1/4}$ (the values of z computed below show that these terms are higher than those deleted)

$$\begin{aligned} v_x &= v_{x0} x^{(3-4\kappa)/6\kappa} (v_{x21} + C_2 x^{1/4} v_{x22} + \dots) \\ v_y &= v_{y0} x^{(3-6\kappa)/6\kappa} (v_{y21} + C_2 x^{1/4} v_{y22} + \dots) \\ p &= p_0 x^{-1/4} p_{21} + \dots \\ \rho &= \rho_0 x^{-1/2\kappa} (\rho_{21} + C_2 x^{1/4} \rho_{22} + \dots) \\ w &= w_0 x^{(3-4\kappa)/6\kappa} (w_{21} + C_2 x^{1/4} w_{22} + \dots) \\ y &= C x^{(\kappa+3)/6\kappa} (y_{21} + C_2 x^{1/4} y_{22} + \dots) \end{aligned} \tag{3.3}$$

The first-approximation function first studied in /1/, can be expressed in terms of special functions /3/. The second-approximation functions satisfy the linear system

$$\begin{aligned} \left(\frac{3}{8} z + \frac{\kappa + 3}{4\kappa}\right) y_{22} - \frac{\zeta}{4} \frac{dy_{22}}{d\zeta} &= \frac{2}{\kappa + 1} v_{y22} \\ u_{22} &= \frac{\kappa + 1}{\kappa - 1} p_{21} \frac{dy_{22}}{d\zeta}, \quad \rho_{21} u_{22} + \rho_{22} w_{21} = 0 \\ \frac{K}{Pr} \frac{d^2 w_{22}}{d\zeta^2} + \zeta \frac{dw_{22}}{d\zeta} + \left(\frac{1}{\kappa} - \frac{3}{2} z\right) w_{22} &= 0, \quad K = \frac{16\kappa p_{21}}{3(\kappa^2 - 1)} \\ K \frac{d^2 v_{x22}}{d\zeta^2} + \zeta \frac{dv_{x22}}{d\zeta} + \left(\frac{4\kappa - 3}{\kappa} - \frac{3}{2} z\right) v_{x22} &= \frac{4(\kappa - 1)}{\kappa} w_{22} \end{aligned} \tag{3.4}$$

The solution of (3.4) must satisfy the symmetry conditions when $\zeta = 0$, and the matching conditions as $\zeta \rightarrow \infty$. We note that the last two equations can be reduced, by replacing the variable $\xi = -Pr \zeta^2/2K$, to the canonical form of a degenerate hypergeometric equation /11/.

4. Symmetric perturbations. Next we shall study the free perturbations in plane parallel hypersonic flow. Until now the constants C_z and z were assumed to be arbitrary. The symmetry conditions enable us to find the constant z . At the same time, the amplitude C_z of the perturbations in the asymptotic $x \rightarrow \infty$ formulation of the problem cannot be found.

We shall begin by considering the problem of hypersonic flow past a symmetric profile at zero angle of attack. In this case the flow pattern will be symmetrical about the profile axis, and hence the perturbations describing how the flow behind the concrete profile differs, as $x \rightarrow \infty$, from the solution of /1/, will also be symmetrical. After determining the symmetrical perturbations we obtain at once the conditions which must be satisfied by the functions of the second and third approximation in the region of the wake

$$\frac{dv_{x22}}{d\xi^2}(0) = v_{y22}(0) = \frac{d\rho_{22}}{d\xi^2}(0) = \frac{dw_{22}}{d\xi^2}(0) = y_{22}(0) = y_{32}(0) = 0 \tag{4.1}$$

The last equations (4.1) and (3.2) together yield an equation for determining the complex constant z

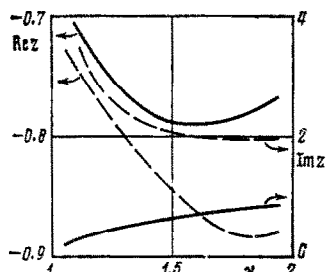
$$c_y(z, \kappa) = 0 \tag{4.2}$$

Equation (4.2) was encountered earlier /12/ in the course of investigating free perturbations of a one-dimensional unsteady flow of gas displaced by a piston. The limiting cases $\kappa \rightarrow 1+0, \kappa \rightarrow \infty$ were studied and the case of a monoatomic gas $\kappa = 5/3$, though only a single pair of complex conjugate roots, which in /12/ was regarded as the roots of smallest modulus, was determined for the latter value of κ .

Below we give the results of a more detailed study of (4.2). A program for the numerical integration of system (2.3) with the Cauchy data (2.4) was written in order to compute the roots of this equation. In the course of solving problem (2.3), (2.4) we differentiated it with respect to z , and then integrated the result. Although the numerical integration was not, for various reasons, carried out to very small values of η , nevertheless the use of three subsequent terms in the asymptotic $\eta \rightarrow 0$ expansions (2.5) for p_{12} and y_{12} gave the constants $c_y, c_p, dc_y/dz$ and dc_p/dz with sufficient accuracy.

Using Newton's method to compute the roots of (4.2) we found that within the range of variation of z , (4.2) has pairwise complex conjugate roots only. For $\kappa = 1.4$ the first five roots of (4.2) are: $-0.779 + i0.582$; $-0.835 + i2.651$; $-0.857 + i4.626$; $-0.863 + i6.594$; $-0.865 + i8.560$. Computing the roots of (4.2) for $\kappa = 5/3$ we found that the root $z = -0.907 + i2.467$ [12] is the second smallest in modulus and the root $z = -0.790 + i0.743$ is the smallest in modulus. The dependence of the real and imaginary part of this root on κ is shown in the figure by the solid line.

We will complete the construction of the symmetrical solution by giving expressions for the second-approximation functions satisfying the conditions of matching the expansions (2.2) and (3.3), and the symmetry conditions (4.1)



$$v_{22} = c_{v2} M\left(\frac{1-z_y}{2}, \frac{1}{2}, -\frac{Pr}{2K} \xi^2\right), y_{22} = c_{y2} \xi M\left(\frac{1-z_y}{2}, \frac{3}{2}, -\frac{Pr}{2K} \xi^2\right) \tag{4.3}$$

$$c_{y2} = \frac{c_{y2} \Gamma(z_y/2)}{\Gamma(1/2)} \left(\frac{Pr}{2K}\right)^{(1-z_y)/2}, c_{v2} = \frac{\kappa + 1}{\kappa - 1} p_{21} c_{y2}$$

The above relations use the Kummer function $M(a, b, \xi)$ [11] which represents the solution of a degenerate hypergeometric equation. A solution of the equation for v_{x22} satisfying the boundary conditions formulated above, can be obtained using the method of varying the constants, just as in /3/. The functions v_{y22}, ρ_{22} are found from (4.3) with help of the algebraic relations.

5. Antisymmetric perturbations. We shall now consider the problem of hypersonic flow past an asymmetric profile. In this case the flow will be described, as $x \rightarrow \infty$, by a set of symmetric and antisymmetric perturbations of the solution /1/. The second and third approximation functions corresponding to antisymmetric perturbations satisfy the following conditions at the centre of the wake:

$$v_{x22}(0) = \frac{dv_{y22}}{d\xi^2}(0) = \rho_{22}(0) = w_{22}(0) = \frac{dy_{22}}{d\xi^2}(0) = p_{22}(0) = 0 \tag{5.1}$$

The last equation of (5.1) and the first formula of (3.2) together yield the following equation for determining the constants z corresponding to the antisymmetric perturbations:

$$c_p(z, \kappa) = 0 \tag{5.2}$$

The first five complex roots of (5.2) have the form $-0.823 + i2.211$; $-0.855 + i4.188$;

$-0.862 + i6.157$; $-0.865 + i8.123$; $-0.866 + i10.088$ ($\kappa = 1.4$). The dependence on κ of the real and imaginary parts of the root with the smallest modulus is shown in the figure by the dashed line.

Equation (5.2) also has two real roots in the range of values of z in question. The first root $z = -1/3$ describes hypersonic flows in which the streamline profile is acted upon by a lift F_y . The flow were studied in detail in /3/ where it was shown that in this case the constant C_z is proportional to the lift and can be determined, provided that F_y is known. The second real root of (5.2) $z = -2/3$ describes the antisymmetric perturbations generated in the basic solution /1/ by changing the variable $\Psi \rightarrow \Psi + \Delta\Psi$ ($\Delta\Psi$ is a constant). Indeed, carrying out this substitution and expanding the flow parameters in a Taylor series up to terms linear in $\Delta\Psi$, we obtain the correction terms of order $x^{-1/2}$.

Analogous perturbations, albeit the symmetric ones, can be obtained by making the substitution $z \rightarrow z + \Delta z$ or $C \rightarrow C + \Delta C$, $\Delta C \ll 1$ in the basic solution /1/. The perturbations with powder index $z = -1$ and $z = 0$ correspond to the first and second substitution respectively. Both values of z satisfy (4.2), but fall outside the range of admissible values of z (in fact, they form the end points of this interval). In all cases we have $z = -1, -2/3, 0$ and the second-approximation functions are expressed in terms of the first-approximation functions and their derivatives.

Let us now write expressions for the second-approximation functions satisfying the conditions of matching with the outer region and the symmetry conditions (5.1)

$$\begin{aligned} w_{22} &= c_{wa} \zeta M \left(1 - \frac{z_y}{2}, \frac{3}{2}, -\frac{Pr}{2K} t^2 \right) \\ v_{22} &= c_{va} M \left(-\frac{z_y}{2}, \frac{1}{2}, -\frac{Pr}{2K} t^2 \right) \\ c_{ya} &= c_2 \frac{\Gamma(1/2 + z_y/2)}{\Gamma(1/2)} \left(\frac{Pr}{2K} \right)^{-z_y/2}, \quad c_{wa} = \frac{Pr z_y P_{21} (\kappa + 1)}{K (\kappa - 1)^2} c_{ya} \end{aligned} \quad (5.3)$$

The functions v_{p22} and ρ_{22} are found from (5.3) using the final relations, and solution of the equation for v_{x22} can be constructed using the method of varying the constants.

In conclusion we note that the region of the wake plays a passive role in forming free perturbations, since the constants z are found, in fact, from the solution of the problem (2.3), (2.4) describing the flow in the outer region.

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